9/10/25

Lecture 8

<u>Last time</u> A fn $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation i.f.f. its a matrix transformation. The matrix for T is given column-wise by

$$[T] = [\underbrace{T(\vec{e}_1) \cdots T(\vec{e}_n)}_{\text{Standard basis vectors}}]$$

$\boxed{\text{Ex}}$ $T: \mathbb{R}^3 \to \mathbb{R}^2$ (is linear)

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{bmatrix}1\\1\end{bmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{bmatrix}2\\7\end{bmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

The matrix of T is $\begin{bmatrix}1 & 2 & 0\\ 1 & 7 & 0\end{bmatrix}$

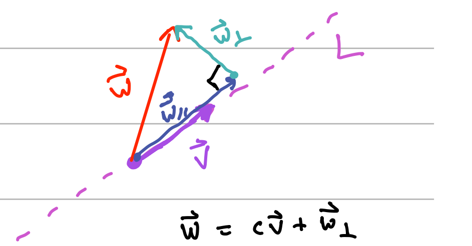
* easily determined the matrix because the given input vectors of T are $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{Standard basis vectors}}$. *

Orthogonal Projection

<u>Theorem</u> → Given a non-$\vec{0}$ $\vec{v} \in \mathbb{R}^n$ let $L = \underbrace{\{c\vec{v} \in \mathbb{R}^n : c \in \mathbb{R}\}}_{\text{this is the line through } \vec{0} + \vec{v}.}$

If $\vec{w} \in \mathbb{R}^n$ then there are unique vectors $\vec{w}_{\parallel}, \vec{w}_{\perp} \in \mathbb{R}^n$ such that:

- $\vec{w} = \vec{w}_{\parallel} + \vec{w}_{\perp}$
- $\vec{w}_{\parallel} \in L$
- $\vec{w}_{\perp}$ is perpendicular to L



$\vec{w} = c\vec{v} + \vec{w}_{\perp}$

$\text{proj}_L(\vec{w}) = \vec{w}_{\parallel}$ = orthogonal projection of $\vec{w}$ onto L

$proj_L = \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

Agenda (for this lecture and part of the next)

Step 0 : What's orthogonal?

Step 1 : Prove Thm + get a formula for $proj_L(\vec{w})$

Step 2 : Show $proj_L$ is linear + find its matrix

Step 0:

Def - the length (or norm) of $\vec{v} \in \mathbb{R}^n$ is the quantity:
$$||v|| = \sqrt{(v_1)^2 + \ldots + (v_n)^2}$$

$\boxed{Ex}$ $v = \langle 1, 2, 3 \rangle$ $\quad ||v|| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}$

Remark: we call $\vec{v} \in \mathbb{R}^n$ a unit vector if $||\vec{v}|| = 1$

- Can scale any non-zero vector $\vec{v} \in \mathbb{R}^n$ to get a unit vector

example $\vec{u} = \dfrac{\vec{v}}{||\vec{v}||}$

3 key properties of length:

- $||\vec{v}|| \geq 0$ and $(||\vec{v}|| = 0$ i.f.f. $\vec{v} = \vec{0})$
- $c \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n \to ||c\vec{v}|| = |c|\,||\vec{v}||$
- Triangle inequality $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$